Aspects of superstring theory in $AdS_5 \times S^5$

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- Reformulation of $AdS_5 \times S^5$ superstring in terms of currents ("Pohlmeyer reduction")

  M. Grigoriev and A.T., arXiv:0711.0155, 0806.2623;

  R. Roiban and A.T., to appear

- 2d dualities of $AdS_5 \times S^5$ superstring and dual superconformal symmetry

  N. Beisert, R. Ricci, A.T. and M. Wolf, arXiv:0807.3228

Major goal:
solve string theory in $AdS_5 \times S^5$
use conformal invariance,
global (super)symmetry and integrability

find S-matrix and justify Bethe Ansatz for the spectrum
from first principles

String Theory in $AdS_5 \times S^5$

bosonic coset $\frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$

generalized to GS string: supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$

(Metsaev, AT 98)

$$S = T \int d^2\sigma \big[G_{mn}(x)\partial x^m \partial x^n + \bar\theta(D + F_5)\theta\partial x$$

$$+ \bar\theta\theta\bar\theta\theta\partial x\partial x + ...\big],$$

tension $T = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}$

Conformal invariance: $\beta_{mn} = R_{mn} - (F_5)^2_{mn} = 0$

Classical integrability of coset σ-model (Luscher,Pohlmeyer 76)

same for classical $AdS_5 \times S^5$ superstringq

(Bena, Polchinski, Roiban 02)

extends to quantum level: 1- and 2-loop computations and their comparison to Bethe ansatz (work of last 5 years)

Green-Schwarz superstring

Superstring in curved type II supergravity background

$\int d^2\sigma\, G_{MN}(Z)\partial Z^M \partial Z^N + \ldots\,, \qquad Z^M = (x^m, \theta^I_\alpha)$

$m = 0, 1, \ldots 9, \quad \alpha = 1, 2\ldots, 16, \quad I = 1, 2$

Explicit form of action is generally hard to find

$AdS_5 \times S^5$: coset space symmetry facilitates explicit construction

Algebraic construction of unique κ-invariant action as in flat space

$R^{1,9} = \frac{G}{H} = \frac{\text{Poincare}}{\text{Lorentz}}$

Flat superspace = $\frac{\widehat{G}}{H}$= $\frac{\text{SuperPoincare}}{\text{Lorentz}}$

structure of action is fixed by superPoincare algebra $(\mathcal{P}, \mathcal{M}, \mathcal{Q})$

$[\mathcal{P}, \mathcal{M}] \sim \mathcal{P}, \;\; [\mathcal{M}, \mathcal{M}] \sim \mathcal{M}, \;\; [\mathcal{M}, \mathcal{Q}] \sim \mathcal{Q}, \;\; \{\mathcal{Q}, \mathcal{Q}\} \sim \mathcal{P}$

$g^{-1}dg = J^m \mathcal{P}_m + J^I_\alpha \mathcal{Q}^\alpha_I + J^{mn}\mathcal{M}_{mn}$

Supercoset action= $\int \text{Tr}(g^{-1}dg)^2_{G/H}$ + fermionic WZ-term

$I = \int d^2\sigma (J^m J^m + a\bar{J}^I J^I) + b\int \; J^m \wedge \bar{J}^I \Gamma_m J^J s_{IJ}$

$s_{IJ} = (1, -1)$

$J^m = dx^m - i\bar\theta^I\Gamma^m\theta^I, \qquad J^I_\alpha = d\theta^I_\alpha$

manifest superPoincare symmetry but

unitarity and right fermionic spectrum iff $a = 0,\ b = \pm 1$:

κ-invariance $\rightarrow$ Green-Schwarz action:

$$L = -\tfrac{1}{2}(\partial_a x^m - i\bar\theta^I\Gamma^m\partial_a\theta^I)^2$$

$$+\ i\epsilon^{ab}s_{IJ}\bar\theta^I\Gamma_m\partial_a\theta^J(\partial_b x^m - \tfrac{i}{2}\bar\theta^K\Gamma^m\partial_b\theta^K)$$

peculiar “degenerate” Lagrangian: no $\partial\bar\theta\partial\theta$ term

$L \sim \partial x\partial x + \partial x\bar\theta\partial\theta + (\bar\theta\partial\theta)^2$

perturbative expansion is well-defined

near $\bar x$ background, e.g., $x^m = N^m_a\sigma^a$

$x = \bar x + \xi,\ \ \theta' = \sqrt{\partial\bar x}\,\theta$

$L \sim \partial\xi\partial\xi + \bar\theta'\partial\theta' + \frac{1}{\sqrt{\partial\bar x}}\partial\xi\bar\theta'\partial\theta' + \ldots$

non-renormalizable by power counting

but κ-symmetry (uniqueness of action) implies finiteness direct check of cancellation of 2-loop logarithmic UV divergences and trivial partition function (Roiban, Tirziu, AT 07)

preservation of κ-symmetry implies that semiclassical loop (α') expansion must be finite also in curved space

regularization issues are non-trivial starting with 2 loops

$AdS_5 \times S^5 = \frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$

Killing vectors and Killing spinors of $AdS_5 \times S^5$:
$PSU(2,2|4)$ symmetry
replace G/H=SuperPoincare/Lorentz in flat GS case by

$$\frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$$

generators: $(\mathcal{P}_q, \mathcal{M}_{pq}); (\mathcal{P}'_r, \mathcal{M}'_{rs}); \mathcal{Q}^I_\alpha, \quad m = (q,r)$

$$[\mathcal{P},\mathcal{P}] \sim \mathcal{M}, \ \ [\mathcal{P},\mathcal{M}] \sim \mathcal{P}, \ \ [\mathcal{M},\mathcal{M}] \sim \mathcal{M},$$
$$[\mathcal{Q},\mathcal{P}_q] \sim \gamma_q \mathcal{Q}, \ \ [\mathcal{Q},\mathcal{M}_{pq}] \sim \gamma_{pq} \mathcal{Q}$$
$$\{\mathcal{Q}^I, \mathcal{Q}^J\} \sim \delta^{IJ}(\gamma \cdot \mathcal{P} + \gamma' \cdot \mathcal{P}') + \epsilon^{IJ}(\gamma \cdot \mathcal{M} + \gamma' \cdot \mathcal{M}')$$

$PSU(2,2|4)$ invariant action:

$\int \mathrm{Tr}(g^{-1}dg)^2_{G/H}$ + WZ-term

$J = g^{-1}dg = J^m \mathcal{P}_m + J^I_\alpha \mathcal{Q}^\alpha_I + J^{mn} \mathcal{M}_{mn}$

$$I = \frac{\sqrt{\lambda}}{2\pi} \Big[\int d^2\sigma (J^m J^m + a \bar{J}^I J^I) + b \int J^m \wedge \bar{J}^I \Gamma_m J^J s_{IJ} \Big]$$

as in flat space $a = 0, \ \ b = \pm 1$ required by κ-symmetry

unique action with right symmetry and right flat-space limit

Formal argument for UV finiteness (2d conformal invariance):

1. global symmetry – only overall coefficient
   of J^2 term (radius) can run
2. non-renormalization of WZ term (homogeneous 3-form)
3. preservation of κ-symmetry at the quantum level
   – relating coefficients of J^2 and WZ terms

Equivalent form of the GS action:

$AdS_5 \times S^5 = \frac{SU(2,2)}{Sp(2,2)} \times \frac{SU(4)}{Sp(4)}$

generalized to

$\frac{\widehat{F}}{G} = \frac{PSU(2,2|4)}{Sp(2,2)\times Sp(4)}$

basic superalgebra $\widehat{\mathfrak{f}} = psu(2,2|4)$

bosonic part $\mathfrak{f} = su(2,2) \oplus su(4) \cong so(2,4) \oplus so(6)$

admits Z_4-grading:

(Berkovits, Bershadsky, Hauer, Zhukov, Zwiebach 89)

$$\widehat{\mathfrak{f}} = \mathfrak{f}_0 \oplus \mathfrak{f}_1 \oplus \mathfrak{f}_2 \oplus \mathfrak{f}_3\,, \qquad\qquad [\mathfrak{f}_i, \mathfrak{f}_j] \subset \mathfrak{f}_{i+j \bmod 4}$$

$\mathfrak{f}_0 = \mathfrak{g} = sp(2,2) \oplus sp(4)$

$\mathfrak{f}_2 = AdS_5 \times S^5$

current $J = f^{-1}\partial_a f,\ \ f \in \widehat{F}$ (notation change!)

$$J_a = f^{-1}\partial_a f = \mathcal{A}_a + Q_{1a} + P_a + Q_{2a}$$

$$\mathcal{A} \in \mathfrak{f}_0, \quad Q_1 \in \mathfrak{f}_1, \quad P \in \mathfrak{f}_2, \quad Q_2 \in \mathfrak{f}_3\,.$$

$\widehat{\mathfrak{f}} = psu(2,2|4)$: quotient of $su(2,2|4)$ by aI

$su(2,2|4)$ as 8×8 matrix algebra

$$M = \begin{pmatrix} A & X \\ X^\dagger\Sigma & B \end{pmatrix}\,, \qquad \mathrm{Tr}A - \mathrm{Tr}B = 0, \qquad A \in u(2,2)\,, \qquad B \in u(4)\,.$$

$$\Sigma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad K = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad [\Sigma, K] = 0$$

Z_4 split: $\quad M = M_0 \oplus M_1 \oplus M_2 \oplus M_3$

$$M_{0,2} = \begin{pmatrix} A_{0,2} = \frac{1}{2}(A \pm KA^tK) & 0 \\ 0 & b_{0,2} = \frac{1}{2}(B \pm KB^tK) \end{pmatrix}\,,$$

$$M_{1,3} = \begin{pmatrix} 0 & X_{1,3} \\ X^\dagger_{1,3}\Sigma & 0 \end{pmatrix}\,, \quad X_{1,3} = \mathcal{P}_\pm X = \frac{1}{2}(X \pm i\Sigma K X^* K).$$

$M_0 \in sp(2,2)\oplus sp(4)$, M_2 in bosonic part of coset; M_1 and M_3 real and imaginary parts of X – “reality decomposition”:

elements from $\widehat{\mathfrak{f}}_1$ and $\widehat{\mathfrak{f}}_3$ satisfy $X^*_{1,3} = \mp i\Sigma K X_{1,3} K$ can be solved in terms of 4×4 *real* Grassmann $\mathcal{X}_{1,3}$

$$X_1 = \mathcal{X}_1 + i\Sigma K \mathcal{X}_1 K\,, \qquad X_3 = \mathcal{X}_3 - i\Sigma K \mathcal{X}_3 K\,.$$

GS Lagrangian:

$$L_{\rm GS} = \frac{1}{2}{\rm STr}(\sqrt{-g}g^{ab}P_aP_b + \varepsilon^{ab}Q_{1a}Q_{2b})\,,$$

simple structure but not standard coset model:

fermionic currents in WZ term only

conformal gauge: $\sqrt{-g}g^{ab} = \eta^{ab}$

$$L_{\rm GS} = {\rm STr}[P_+P_- + \frac{1}{2}(Q_{1+}Q_{2-} - Q_{1-}Q_{2+})]$$

$$ {\rm STr}(P_+P_+) = 0\,, \qquad\qquad {\rm STr}(P_-P_-) = 0$$

Equations of motion in terms of currents (1-st order form)

$$\begin{aligned}
{\rm EOM}: \quad & \partial_+P_- + [\mathcal{A}_+, P_-] + [Q_{2+}, Q_{2-}] = 0\,, \\
& \partial_-P_+ + [\mathcal{A}_-, P_+] + [Q_{1-}, Q_{1+}] = 0\,, \\
& [P_+, Q_{1-}] = 0\,, \qquad [P_-, Q_{2+}] = 0\,. \\
{\rm MC}: \quad & \partial_-J_+ - \partial_+J_- + [J_-, J_+] = 0\,.
\end{aligned}$$

κ-gauge condition: $Q_{1-} = 0\,, \qquad Q_{2+} = 0$

remaining EOM:

$$\partial_+ P_- + [\mathcal{A}_+, P_-] = 0\,, \qquad \partial_- P_+ + [\mathcal{A}_-, P_+] = 0$$

Maurer-Cartan:

$$\begin{aligned}
\partial_+ \mathcal{A}_- - \partial_- \mathcal{A}_+ + [\mathcal{A}_+, \mathcal{A}_-] + [P_+, P_-] + [\mathcal{Q}_{1+}, \mathcal{Q}_{2-}] = 0\,,\\
\partial_- \mathcal{Q}_{1+} + [\mathcal{A}_-, \mathcal{Q}_{1+}] - [P_+, \mathcal{Q}_{2-}] = 0\,,\\
\partial_+ \mathcal{Q}_{2-} + [\mathcal{A}_+, \mathcal{Q}_{2-}] - [P_-, \mathcal{Q}_{1+}] = 0\,.
\end{aligned}$$

How to solve quantum string theory in $AdS_5 \times S^5$?

GS string on supercoset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$
not of known solvable type (cf. free oscillators; WZW)
analogy with exact solution of $O(n)$ model (Zamolodchikovs) or principal chiral model (Polyakov-Wiegmann ...) ?
but 2d CFT – no mass generation

By analogy with flat space –
light-cone gauge: analog of $x^+ = p^+\tau,\ p^+ = \text{const},\ \Gamma^+\theta = 0$
Two natural options:
(i) null geodesic parallel to the boundary in Poincare patch –
action/Hamiltonian quartic in fermions (Metsaev, Thorn, AT, 01)
(ii) null geodesic wrapping S^5:
complicated action (Callan et al, 03;
Arutyunov, Frolov, Plefka, Zamaklar, 05-06)

Common problem:

lack of manifest 2d Lorentz symmetry

hard to apply known 2d integrable field theory methods –

S-matrix depends on two rapidities, not on their difference

constraints on it are a priori unclear...

An alternative approach: “Pohlmeyer reduction”

conformal gauge, solve Virasoro conditions

find “reduced” action in terms of currents

use it as a starting point for quantization

Aim: PR version for $AdS_5 \times S^5$ superstring

(i) introduce new fields locally related to supercoset currents

(ii) solve Virasoro condition explicitly

(iii) find local 2d Lorentz-invariant

action for independent (8B+8F) d.o.f

$\rightarrow$ fermionic generalization of non-abelian Toda theory

PR: a nonlocal map that preserves integrable structure

1. gauge-equivalent Lax pairs; map between soliton solutions gives integrable massive local field theory

2. quantum equivalence to original GS model ?

may expect for full $AdS_5 \times S^5$ string model = CFT

3. integrable theory: semiclassical solitonic spectrum may essentially determine quantum spectrum

the two solitonic S-matrices should be closely related:

Lorentz-invariant S-matrix of PR-model should lead to effective magnon S-matrix

Pohlmeyer reduction: bosonic coset models

Prototypical example: S^2-sigma model $\rightarrow$ Sine-Gordon theory

$$L = \partial_+ X^m \partial_- X^m - \Lambda (X^m X^m - 1) \,, \qquad m = 1, 2, 3$$

Equations of motion:

$$\partial_+ \partial_- X^m + \Lambda X^m = 0 \,, \quad \Lambda = \partial_+ X^m \partial_- X^m \,, \quad X^m X^m = 1$$

Stress tensor: $\mathrm{T}_{\pm\pm} = \partial_\pm X^m \partial_\pm X^m$

$$\mathrm{T}_{+-} = 0 \,, \quad \partial_+ \mathrm{T}_{--} = 0 \,, \qquad \partial_- \mathrm{T}_{++} = 0$$

implies $\mathrm{T}_{++} = f(\sigma_+)$, $\mathrm{T}_{--} = h(\sigma_-)$

using the conformal transformations $\sigma_\pm \rightarrow F_\pm(\sigma_\pm)$ can set

$$\partial_+ X^m \partial_+ X^m = \mu^2 \,, \qquad \partial_- X^m \partial_- X^m = \mu^2 \,, \qquad \mu = \text{const} \ .$$

3 unit vectors in 3-dimensional Euclidean space:

$$X^m \,, \qquad X^m_+ = \mu^{-1} \partial_+ X^m \,, \qquad X^m_- = \mu^{-1} \partial_- X^m \,,$$

X^m is orthogonal ($X^m \partial_\pm X^m = 0$) to both X_+^m and X_-^m
remaining $SO(3)$ invariant quantity is scalar product

$$\partial_+ X^m \partial_- X^m = \mu^2 \cos 2\varphi$$

then $\partial_+ \partial_- \varphi + \frac{\mu^2}{2} \sin 2\varphi = 0$
following from sine-Gordon action (Pohlmeyer, 1976)

$$\widetilde{L} = \partial_+ \varphi \partial_- \varphi + \frac{\mu^2}{2} \cos 2\varphi$$

2d Lorentz invariant despite explicit constraints
Classical solutions and integrable structures
(Lax pair, Backlund transformations, etc) are directly related
e.g., SG soliton mapped into rotating folded string on S^2
"giant magnon" in the $J = \infty$ limit (Hofman, Maldacena 06)

Analogous construction for S^3 model gives

Complex sine-Gordon model (Pohlmeyer; Lund, Regge 76)

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \cot^2\varphi\,\partial_+\theta\partial_-\theta + \frac{\mu^2}{2}\cos 2\varphi$$

φ, θ are $SO(4)$-invariants:

$$\mu^2 \cos 2\varphi = \partial_+ X^m \partial_- X^m$$

$$\mu^3 \sin^2\varphi\,\partial_\pm\theta = \mp\frac{1}{2}\epsilon_{mnkl}X^m\partial_+X^n\partial_-X^k\partial^2_\pm X^l$$

"String on $R_t \times S^n$" interpretation

conformal gauge plus $t = \mu\tau$ to fix conformal diffeomorphisms:

$\partial_\pm X^m \partial_\pm X^m = \mu^2$ are Virasoro constraints

Similar construction for AdS_n case,

i.e. string on $AdS_n \times S^1_\psi$ with $\psi = \mu\tau$

e.g. reduced theory for $AdS_3 \times S^1$

$$\widetilde{L} = \partial_+\phi\partial_-\phi + \coth^2\varphi\,\partial_+\chi\partial_-\chi - \frac{\mu^2}{2}\cosh 2\phi$$

Comments:

- Virasoro constraints are solved by a special choice of variables related nonlocally to the original coordinates
- Although the reduction is not explicitly Lorentz invariant the resulting Lagrangian turns out to be 2d Lorentz invariant
- The reduced theory is formulated in terms of manifestly $SO(n)$ invariant variables: "blind" to original global symmetry
- reduced theory is equivalent to the original theory as integrable system: the respective Lax pairs are gauge-equivalent
- PR may be thought of as a formulation in terms of physical d.o.f. – coset space analog of flat-space l.c. gauge (where 2d Lorentz is unbroken)
- in S^n case reduced theory can not be quantum-equivalent to the original one (e.g., conformal symmetry was assumed in the reduction procedure)

PR for bosonic F/G-coset model

G/H gauged WZW model + relevant integrable potential

F/G-coset sigma model: symmetric space

$$\mathfrak{f} = \mathfrak{p} \oplus \mathfrak{g}\,, \qquad [\mathfrak{g},\mathfrak{g}] \subset \mathfrak{g}\,, \qquad [\mathfrak{g},\mathfrak{p}] \subset \mathfrak{p}\,, \qquad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{g}$$

$$J = f^{-1}df = \mathcal{A} + P\,, \qquad \mathcal{A} = J_{\mathfrak{g}} \in \mathfrak{g}\,, \quad P = J_{\mathfrak{p}} \in \mathfrak{p}\,.$$

$$L = -\mathrm{Tr}(P_+ P_-)$$

G gauge transformations $f \to fg$;

global F-symmetry: $f \to f_0 f$, $f_0 = \mathrm{const} \in F$

classical conformal invariance

$J = \mathcal{A} + P$ as fundamental variables

$D_+P_- = 0\,, \quad D_-P_+ = 0\,, \qquad D = d + [\mathcal{A},\ \ ]$ – EOM

$D_-P_+ - D_+P_- + [P_+, P_-] + \mathcal{F}_{+-} = 0$ – Maurer-Cartan

$\mathrm{Tr}(P_+P_+) = -\mu^2\,, \quad \mathrm{Tr}(P_-P_-) = -\mu^2$ – Virasoro

Main idea: – first solve EOM and Virasoro and then MC
special choice of G gauge condition and conformal diffs. $\rightarrow$
find reduced action giving eqs. resulting from MC
gauge fixing that solves the first Virasoro constraint

$$P_+ = \mu\, T = \text{const}\,, \qquad T \in \mathfrak{p} = \mathfrak{f} \ominus \mathfrak{g}, \qquad \mathrm{Tr}(TT) = -1$$

choice of special element $T \rightarrow$ decomposition of the algebra of F

$$\mathfrak{f} = \mathfrak{p} \oplus \mathfrak{g}\,, \qquad \mathfrak{p} = T \oplus \mathfrak{n}\,, \qquad \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}\,, \qquad [T, \mathfrak{h}] = 0\,,$$

$\mathfrak{h}$ is a centraliser of T in $\mathfrak{g}$

EOM $D_- P_+ = 0$ is solved by $\mathcal{A}_- = (\mathcal{A}_-)_{\mathfrak{h}} \equiv A_-$

second Virasoro constraint is solved by

$$P_- = \mu \, g^{-1} T g \, , \qquad g \in G$$

EOM $D_+ P_- = 0$ is solved by $\mathcal{A}_+ = g^{-1} \partial_+ g + g^{-1} A_+ g$

To summarise: new dynamical field variables

G-valued field g , $\mathfrak{h}$-valued fields A_+, A_-, $[T, A_\pm] = 0$

Relation to G/H gauged WZW model

remaining Maurer-Cartan equation on $g, A_\pm$ follows from
G/H gWZW action with potential:

$$
\begin{aligned}
L = \quad & - \quad \frac{1}{2}\mathrm{Tr}(g^{-1}\partial_+ g g^{-1}\partial_- g) + \text{WZ term} \\
& - \quad \mathrm{Tr}\Big(A_+\,\partial_- g g^{-1} - A_-\, g^{-1}\partial_+ g - g^{-1}A_+ g A_- + A_+ A_-\Big) \\
& - \quad \mu^2 \mathrm{Tr}(T g^{-1} T g)
\end{aligned}
$$

Pohlmeyer-reduced theory for F/G coset sigma model
(Bakas, Park, Shin 95; Grigoriev, AT 07)
reduced theory for strings on $R_t \times F/G$ or $F/G \times S^1_\psi$
integrable potential: relation at the level of Lax pairs
special case of non-abelian Toda theory:
"symmetric space Sine-Gordon model"
(Hollowood, Miramontes et al 96)

A_+, A_-: integrate out or gauge-fix

Reduced equation of motion in the "on-shell" gauge $A_\pm = 0$:

Non-abelian Toda equations:

$$\partial_-(g^{-1}\partial_+ g) - \mu^2[T, g^{-1}Tg] = 0\ ,$$
$$(g^{-1}\partial_+ g)_{\mathfrak{h}} = 0\ , \qquad\qquad (\partial_- g g^{-1})_{\mathfrak{h}} = 0\ .$$

$F/G = SO(n+1)/SO(n) = S^n$: $G/H = SO(n)/SO(n-1)$

$$g = \begin{pmatrix} k_1 & k_2 & \dots & k_n \\ \dots & \dots & \dots & \dots \end{pmatrix}\ , \qquad \sum_{1=1}^{n} k_l k_l = 1$$

get (in general non-Lagrangian) EOM for k_m

$$\partial_-\Big(\frac{\partial_+ k_\ell}{\sqrt{1-\sum_{m=2}^{n} k_m k_m}}\Big) = -\mu^2 k_\ell\ , \qquad \ell = 2, \dots, n\ .$$

Linearising around the vacuum $g = 1$ (i.e. $k_1 = 1,\ k_\ell = 0$)

$$\partial_+\partial_- k_\ell + \mu^2 k_\ell + O(k_\ell^2) = 0$$

massive spectrum: non-trivial S-matrix with H global symmetry

$F/G = SO(n+1)/SO(n) = S^n$:

parametrization of g in Euler angles

$g = e^{T_{n-2}\theta_{n-2}} ... e^{T_1\theta_1} e^{2T\varphi} e^{T_1\theta_1} ... e^{T_{n-2}\theta_{n-2}}$

and integrating out $H = SO(n-1)$ gauge field $A_\pm$

leads to reduced theory that generalizes SG and CSG

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + G_{pq}(\varphi,\theta)\partial_+\theta^p\partial_-\theta^q + \frac{\mu^2}{2}\cos 2\varphi$$

gWZW for $G/H = SO(n)/SO(n-1)$

$$ds^2_{n=2} = d\varphi^2 \ , \qquad\qquad ds^2_{n=3} = d\varphi^2 + \cot^2\varphi\, d\theta^2$$

$G/H = SO(5)/SO(4)$:

$$ds^2_{n=4} = d\varphi^2 + \cot^2\varphi\,(d\theta_1 + \cot\theta_1 \tan\theta_2\, d\theta_2)^2 + \tan^2\varphi\,\frac{d\theta_2^2}{\sin^2\theta_1}$$

and similar for $G/H = S^5 = SO(6)/SO(5)$

Bosonic strings on $AdS_n \times S^n$

straightforward generalization:

Lagrangian and the Virasoro constraints

$L = \mathrm{Tr}(P_+^A P_-^A) - \mathrm{Tr}(P_+^S P_-^S)\,,$

$\mathrm{Tr}(P_\pm^S P_\pm^S) - \mathrm{Tr}(P_\pm^A P_\pm^A) = 0$

fix conformal symmetry by

$\mathrm{Tr}(P_\pm^S P_\pm^S) = \mathrm{Tr}(P_\pm^A P_\pm^A) = -\mu^2$

then PR applies independently in each sector:

get direct sum of reduced systems for S^n and AdS_n

linked by Virasoro, i.e. common μ

e.g. for $F/G = AdS_2 \times S^2$:

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \partial_+\phi\partial_-\phi + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi)$$

Reduced theory for $AdS_5 \times S^5$ superstring

$AdS_5 \times S^5 = \frac{SU(2,2)}{Sp(2,2)} \times \frac{SU(4)}{Sp(4)}$

$$L_{\rm GS} = {\rm STr}[P_+P_- + \frac{1}{2}(Q_{1+}Q_{2-} - Q_{1-}Q_{2+})]$$

$${\rm STr}(P_+P_+) = 0\,, \qquad\qquad {\rm STr}(P_-P_-) = 0$$

PR procedure: solve first EOM and Virasoro

κ-gauge condition: $Q_{1-} = 0\,, \qquad Q_{2+} = 0$

as in bosonic F/G case fix the “reduction gauge”

$$P_+ = \mu\, T\,,$$

$$T = \frac{i}{2}{\rm diag}(1,1,-1,-1|1,1,-1,-1)\,,$$

$$P_- = \mu\, g^{-1}Tg\,, \quad \mathcal{A}_+ = g^{-1}\partial_+ g + g^{-1}A_+ g\,, \quad \mathcal{A}_- = A_-$$

T defines H or $\mathfrak{h}$ by $[\mathfrak{h}, T] = 0$:

$\mathfrak{h} = su(2) \oplus su(2) \oplus su(2) \oplus su(2)$

new variables:

$$g = \begin{pmatrix} g_a & 0 \\ 0 & g_s \end{pmatrix} \,, \qquad g_a \in Sp(2,2), \quad g_s \in Sp(4)$$

$\mathfrak{h} = [su(2)]^4$-valued field $A_\pm$

AdS_5 and S^5 sectors now coupled by fermions

$$\Psi_{_R} \equiv \frac{1}{\sqrt{\mu}} Q_{1+} \,, \qquad \Psi_{_L} \equiv \frac{1}{\sqrt{\mu}} \, g Q_{2-} g^{-1} \,.$$

fix residual κ-symmetry using T:

$$\Psi_{_{R,L}} = \Psi^{\|}_{R,L} \,, \qquad \Psi^{\|}_{_{R,L}} T = -T \Psi^{\|}_{_{R,L}}$$

Fermions link bosons from $Sp(2,2)$ and $Sp(4)$ transforming under both groups parametrization of $\Psi_{R,L}$ in terms of 4 real Grassmann 2×2 matrices $\xi_{R,L}$ and $\eta_{R,L}$

$$\Psi_{R,L} = \begin{pmatrix} 0 & 0 & 0 & \alpha_{R,L} \\ 0 & 0 & \beta_{R,L} & 0 \\ 0 & -\beta^{\dagger}_{R,L} & 0 & 0 \\ \alpha^{\dagger}_{R,L} & 0 & 0 & 0 \end{pmatrix}$$

$$\alpha_{R,L} = \xi_{R,L} + iJ\xi_{R,L}J\,, \qquad \beta_{R,L} = \eta_{R,L} - iJ\eta_{R,L}J$$

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Reduced action for $AdS_5 \times S^5$ superstring

(Grigoriev, AT 07; Mikhailov, Schafer-Nameki 07)

classical gauge-fixed 1-st order equations in terms of currents follow from an action!

fermionic generalization of "gWZW+ potential" theory for

$$\frac{G}{H} = \frac{Sp(2,2)}{SU(2)\times SU(2)} \times \frac{Sp(4)}{SU(2)\times SU(2)}$$

$$\begin{aligned} L &= L_{\rm gWZW}(g, A_+, A_-) + \mu^2\, {\rm STr}(g^{-1}TgT) \\ &+ {\rm STr}\left(\Psi_{_L} T D_+ \Psi_{_L} + \Psi_{_R} T D_- \Psi_{_R}\right) \\ &+ \mu\, {\rm STr}\left(g^{-1}\Psi_{_L} g \Psi_{_R}\right) \end{aligned}$$

sum of PR theories for AdS_5 and S^5 "glued" by fermions

$$\begin{aligned} L &= \widetilde{L}_{AdS_5}(g_a, A_{\pm,a}) + \widetilde{L}_{S^5}(g_s, A_{\pm,s}) \\ &+ \psi_{_L} D_+ \psi_{_L} + \psi_{_R} D_+ \psi_{_R} + O(\mu) \end{aligned}$$

similar but not same as susy gWZW:

fermions are in “mixed” representation

standard 2d kin. terms

$$
\begin{aligned}
L_F &= \mathrm{STr}(\Psi_{_L} T \partial_+ \Psi_{_L} + \Psi_{_R} T \partial_- \Psi_{_R}) + \ldots \\
&= -2i\ \mathrm{Tr}(\xi^t_{_L} \partial_+ \xi_{_L} + \eta^t_{_L} \partial_+ \eta_{_L} + \xi^t_{_R} \partial_- \xi_{_R} + \eta^t_{_R} \partial_- \eta_{_R}) + \ldots
\end{aligned}
$$

integrable model: Lax pair encoding equations of motion

$$
\begin{aligned}
\mathcal{L}_- &= \partial_- + A_- + \ell^{-1} \sqrt{\mu} g^{-1} \Psi_{_L} g + \ell^{-2} \mu g^{-1} T g \,, \\
\mathcal{L}_+ &= \partial_+ + g^{-1} \partial_+ g + g^{-1} A_+ g + \ell \sqrt{\mu} \Psi_{_R} + \ell^2 \mu T
\end{aligned}
$$

Comments:

- gWZW model coupled to the fermions interacting minimally and through the "Yukawa term"
- 2d Lorentz invariant with Ψ_R, Ψ_L as 2d Majorana spinors
- 8 real bosonic and 16 real fermionic independent variables
- 2d supersymmetry? yes, in $AdS_2 \times S^2$ case: $n = 2$ super sine-Gordon
- μ-dependent interactions are equal to GS Lagrangian; gWZW produces MC eq.: path integral derivation via change from fields to currents?
- quadratic in fermions (like susy version of gWZW); integrating out $A_\pm$ gives quartic fermionic terms (reflecting curvature)
- linearisation of EOM in the gauge $A_\pm = 0$ around $g = \mathbf{1}$ describes 8+8 massive bosonic and fermionic d.o.f. with mass μ: same as in BMN limit
- symmetry of resulting relativistic S-matrix: $H = [SU(2)]^4$ – as bosonic part of magnon S-matrix symmetry $[PSU(2|2)]^2$

Example: superstring on $AdS_2 \times S^2$

$$T = \frac{i}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad A_\pm = 0$$

$$g = \begin{pmatrix} \cosh\phi & \sinh\phi & 0 & 0 \\ \sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & \cos\varphi & i\sin\varphi \\ 0 & 0 & i\sin\varphi & \cos\varphi \end{pmatrix} \in SO(1,1) \times SO(2)$$

$$\Psi_R = \begin{pmatrix} 0 & 0 & 0 & i\gamma \\ 0 & 0 & -\beta & 0 \\ 0 & i\beta & 0 & 0 \\ \gamma & 0 & 0 & 0 \end{pmatrix}, \quad \Psi_L = \begin{pmatrix} 0 & 0 & 0 & \rho \\ 0 & 0 & -i\nu & 0 \\ 0 & \nu & 0 & 0 \\ i\rho & 0 & 0 & 0 \end{pmatrix}$$

PR Lagrangian: same as $n=2$ supersymmetric sine-Gordon!

$$\widetilde{L} = \partial_+\varphi\partial_-\varphi + \partial_+\phi\partial_-\phi + \frac{\mu^2}{2}(\cos 2\varphi - \cosh 2\phi) \\ + \beta\partial_-\beta + \gamma\partial_-\gamma + \nu\partial_+\nu + \rho\partial_+\rho \\ - 2\mu\left[\cosh\phi\ \cos\varphi\,(\beta\nu+\gamma\rho) + \sinh\phi\ \sin\varphi\,(\beta\rho-\gamma\nu)\right] .$$

indeed, equivalent to

$$\widetilde{L} = \partial_+\Phi\partial_-\Phi^* - |W'(\Phi)|^2 \\ +\psi^*_{L}\partial_+\psi_{L} + \psi^*_{R}\partial_-\psi_{R} + \left[W''(\Phi)\psi_{L}\psi_{R} + W^{*\prime\prime}(\Phi^*)\psi^*_{L}\psi^*_{R}\right] .$$

bosonic part is of $AdS_2 \times S^2$ bosonic reduced model if

$$W(\Phi) = \mu\cos\Phi\ , \qquad |W'(\Phi)|^2 = \frac{\mu^2}{2}(\cosh 2\phi - \cos 2\varphi)\ .$$

$$\psi_{L} = \nu + i\rho\,, \qquad \psi_{R} = -\beta + i\gamma\ ,$$

UV finiteness of the reduced theory

(R. Roiban, A.T., to appear)

Reduction procedure may work at the quantum level only
in conformally invariant case (as should be in $AdS_5 \times S^5$ case)
Consistency requires that reduced theory is also UV finite
gWZW+ free fermions is finite,
μ-dependent terms may renormalize
fermions should cancel bosonic renormalization
indeed true in $AdS_2 \times S^2$ case ($n = 2$ sine-Gordon)
true also in general:

$$\mathrm{STr}(g^{-1}TgT) = \mathrm{Tr}(g_a^{-1}Tg_aT) - \mathrm{Tr}(g_s^{-1}TgT_s)$$
$$\rightarrow \cos 2\varphi - \cosh 2\phi$$

$\cos 2\varphi$ is “relevant”, $\cosh 2\phi$ - “irrelevant”
bosonic 1-loop correction $\sim (\cos 2\varphi + \cosh 2\phi)$
but fermions cancel this divergence

directly verified at 1-loop and 2-loop order
compute effective action $\Gamma[g]$
after first "rotating away" gauge field $A_\pm$:

$$I_{G/H}[g, A] = I_G[h^{-1}gh'] - I_H[h^{-1}h']$$
$$A_+ = h^{-1}\partial_+ h, \quad A_+ = h'^{-1}\partial_+ h'$$

possible divergences:
$\sim \mathrm{Tr}(g^{-1}TgT)$ at odd loops, $\sim \mathrm{STr}(g^{-1}TgT)$ at even loops
but cancel order by order between bosons and fermions
Thus μ is not renormalized, remains an arbitrary
conformal symmetry gauge fixing parameter at quantum level
In contrast to l.c. gauge fixed GS superstring
the reduced model is 2d Lorentz invariant
and power counting renormalizable (finite).
Classically integrable; prove integrability at the quantum level?

Open questions

- Quantum equivalence of reduced theory and GS theory?

  Path integral argument of equivalence?

  Potential terms is original action

  $\mathrm{Tr}(P_+P_-) = \mu^2 \mathrm{Tr}(Tg^{-1}Tg)$ and same for Yukawa

  gWZW term from change of variables ?

  Rough idea: string in $R_t \times F/G$

  $L = -(\partial t)^2 + \mathrm{Tr}(f^{-1}df + B)^2 \,, \quad f \in F, \;\; B \in \mathfrak{g}$

  string path integral in conformal and $t = \mu\tau$ gauge:

$$\int DfDB \, \delta(T_{++} - \mu^2) \, \delta(T_{--} - \mu^2) \quad e^{iI(f,B)}$$

replace $f^{-1}df$ by C

$$\int DCDBDv\, \delta(T_{++} - \mu^2)\delta(T_{--} - \mu^2)$$
$$\times\exp[i\int (C+B)^2 + v(dC + C\wedge C)]$$

set $(C+B)_+ = \mu T$, $(C+B)_- = \mu g^{-1}Tg$; change from C, B, v to $g \in G, A \in \mathfrak{h}$: $[\mathfrak{h}, T] = 0$

Transformation may work only in genuine quantum-conformal ($AdS_5 \times S^5$) case.

- Indication of equivalence: semiclassical expansion

  near analog of (S, J) rigid string in $AdS_5 \times S^5$ leads to same characteristic frequencies

  – same 1-loop partition function (Roiban, AT 08)

- Tree-level S-matrix for elementary excitations?

  Manifest $SU(2) \times SU(2) \times SU(2) \times SU(2)$ symmetry?

  Relation to magnon S-matrix in BA?

2d dualities of $AdS_5 \times S^5$ string and dual superconformal symmetry

(Beisert, Ricci, AT, Wolf 08)

General remarks:

scalar 2d duality $dx \to *d\tilde{x}$ or “T-duality”

$(\partial y)^2 + G(y)(\partial x)^2 \ \to \ (\partial y)^2 + G^{-1}(y)(\partial \tilde{x})^2$

symmetry of 1-st order (phase-space) equations

but in general changes global symmetry of sigma model

i.e. of the metric “seen” by point particle

($dy^2 + \sin^2 y\, dx^2 \ \to \ dy^2 + \sin^{-2} y\, d\tilde{x}^2, \ SO(3) \to SO(2)$)

thus changes set of conserved local Noether charges

yet is a symmetry of 2d equations –

conserved charges should not disappear

but may become non-local or hidden

Peculiarity of AdS_n metric in Poincare coordinates:
$dy^2 + e^{2y} dx_m dx_m \ \rightarrow \ dy^2 + e^{-2y} d\tilde{x}_m d\tilde{x}_m$
mapped into same metric up to $y \to -y$
Used to simplify form of GS $AdS_5 \times S^5$ action (Kallosh, AT 98)
and to relate amplitudes to Wilson loops at strong coupling
(Alday, Maldacena 07)
$SO(n-1,2)$ sets of local Noether charges before and after duality
some local charges become non-local and some dual local charges
originate from hidden conserved charges of original model
(Ricci, AT, Wolf 07)
interplay of integrability and global symmetry –
no “doubling” of hidden charges:
Lax conections of original and dual model are equivalent
Relation to dual conformal symmetry at weak coupling
(Drummond, Henn, Korchemsky, Sokatchev 07)

Generalization to $AdS_5 \times S^5$ superstring action:
to map superstring action after duality into itself and thus
get superconformal $PSU(2,2|4)$ symmetry in dual model
one needs to apply 2d duality also to some fermionic coordinates
(Berkovits, Maldacena 08)

The reason behind:
to get a symmetry of 1-st order superstring equations
one needs to transform both bosonic and fermionic currents –
get symmetry of Lax connection and thus of 1-st order system:
original and dual Lax pairs are related by
an automorphism of $psu(2,2|4)$
[also symmetry of string action modulo choice of coset
representative, κ-symmetry gauge choice, analytic continuation]

Noether charges of original model in terms of the dual variables
give possibly non-local conserved charges in the dual model
Existence of additional set of conserved Noether charges
in dual model which are local in dual variables and thus
non-local in the original variables means they must originate
from some hidden conserved charges in original model
The existence of dual superconformal symmetry thus
closely related to integrability of $AdS_5 \times S^5$ superstring.
1-st order system may admit other symmetry transformations
but this "T-duality" is special in that it preserves maximal
possible global symmetry.
Its existence is rooted in structure of superconformal algebra:
possibility to choose translations ($[P_a, P_b] = 0$) and $N = 4$
Poincaré supersymmetries ($\{Q^{i\alpha}, Q^{j\beta}\} = 0,\ [Q, P] = 0$)
as maximal abelian subalgebra in $psu(2, 2|4)$:
$2d$ duality acts on associated 4 b and 8 f string coordinates

To relate it to dual superconformal symmetry of gauge theory
(of Drummond, Henn, Korchemsky, Sokatchev 08)
combine duality action on the “bulk” string coordinates
with action on the vertex operators inserted at the boundary

Bosonic G/H Coset Model

G/H symmetric space coset model: $\mathfrak{g} = \mathfrak{g}_{(0)} + \mathfrak{g}_{(2)} \equiv \mathfrak{h} + \mathfrak{g}_{(2)}$

$$L = \tfrac{1}{2}\mathrm{tr}(j_{(2)} \wedge *j_{(2)}), \qquad j = g^{-1}\mathrm{d}g = j_{(0)} + j_{(2)} \equiv A + j_{(2)}$$

first-order system $(\nabla = \mathrm{d} + A)$

$$\mathrm{d}A + A \wedge A + j_{(2)} \wedge j_{(2)} = 0, \qquad \nabla j_{(2)} = 0\,; \qquad \nabla{*j_{(2)}} = 0$$

follows from flatness of Lax connection

$$\mathrm{j}(z) = A + a j_{(2)} + b{*j_{(2)}}, \quad a, b = \tfrac{1}{2}(z^2 \pm z^{-2})$$

observe formal duality symmetry of this phase space system and its integrable structure

$$j_{(2)} \mapsto \mathrm{i}{*j_{(2)}}, \qquad z \mapsto \mathrm{e}^{\frac{\pi}{4}\mathrm{i}} z$$

To relate coset fields, may define a non-local map
$g \mapsto \widetilde{g}$: $(g^{-1}\mathrm{d}g)_{(2)} = *(\widetilde{g}^{-1}\mathrm{d}\widetilde{g})_{(2)}$

May also consider an analog of non-Abelian duality
in principal chiral model by adding MC eqs with Lagrange
multipliers and integrating over currents in path integral
In general, “dualities” are linear transformations of currents
that map 1st-order system into itself and respect integrability
The T-duality in the case of AdS_n or $AdS_5 \cong \frac{SO(2,4)}{SO(1,4)}$
is special being “self-duality”:
maps the system into one with same global symmetry

$AdS_5 \times S^5$ superstring

$G/H = PSU(2,2|4)/[SO(1,4) \times SO(5)]$

Z_4 grading of $psu(2,2|4)$ implies (notation change!)

$$j \;=\; g^{-1}\mathrm{d}g \;=\; j_{(0)} + j_{(1)} + j_{(2)} + j_{(3)}, \quad j_{(0)} \;\equiv\; A$$

$$S \;=\; \int \text{Str}\left[j_{(2)} \wedge *j_{(2)} + j_{(1)} \wedge j_{(3)}\right],$$

1-st order system: $\mathrm{d}j + j \wedge j \;=\; 0$ + eqs. of motion

$$\begin{aligned}
\mathrm{d}A + A \wedge A + j_{(1)} \wedge j_{(3)} + j_{(2)} \wedge j_{(2)} + j_{(3)} \wedge j_{(1)} \;&=\; 0,\\
\nabla j_{(1)} + j_{(2)} \wedge j_{(3)} + j_{(3)} \wedge j_{(2)} \;&=\; 0,\\
\nabla j_{(2)} + j_{(1)} \wedge j_{(1)} + j_{(3)} \wedge j_{(3)} \;&=\; 0,\\
\nabla j_{(3)} + j_{(1)} \wedge j_{(2)} + j_{(2)} \wedge j_{(1)} \;&=\; 0,
\end{aligned}$$

$$\begin{aligned}
\nabla *j_{(2)} + j_{(3)} \wedge j_{(3)} - j_{(1)} \wedge j_{(1)} \;&=\; 0,\\
[j_{(2)}, \wedge(j_{(1)} + *j_{(1)})] \;&=\; 0,\\
[j_{(2)}, \wedge(j_{(3)} - *j_{(3)})] \;&=\; 0.
\end{aligned}$$

Implied by $\mathrm{dj}(z) + \mathrm{j}(z) \wedge \mathrm{j}(z) \;=\; 0$ for Lax family of flat currents

$$\mathrm{j}(z) \;=\; A + z\, j_{(1)} + \tfrac{1}{2}(z^2 + z^{-2})\, j_{(2)} + z^{-1}\, j_{(3)} + \tfrac{1}{2}(z^2 - z^{-2}) *j_{(2)}$$

Explicit form depends on:
(i) bosonic H-gauge or choice of coset representative
(ii) fermionic κ-symmetry gauge
2d diffeomorphisms not fixed

Standard choice of the superconformal algebra basis
adapted to the Poincaré parametrization of AdS_5
natural for comparison with boundary conformal theory in $R^{1,3}$

$$psu(2,2|4) \;=\; \left\{P_a, L_{ab}, K_a, D, R_i{}^j \,|\, Q^{i\alpha}, \bar{Q}_i^{\dot{\alpha}}, S_i^{\alpha}, \bar{S}^{i\dot{\alpha}}\right\}$$

$a, b = 0, \ldots, 3,\;\; \alpha, \beta = 1, 2,\;\; i, j = 1, \ldots, 4$

Z_4-splitting of $psu(2,2|4) = \mathfrak{h} \oplus \mathfrak{g}_{(1)} \oplus \mathfrak{g}_{(2)} \oplus \mathfrak{g}_{(3)}$

$$\begin{aligned}
\mathfrak{h} &= \left\{ \tfrac{1}{2}(P_a - K_a), L_{ab}, R_{(ij)} \right\}, \\
\mathfrak{g}_{(1)} &= \left\{ \tfrac{1}{2}(Q^{i\alpha} + S^{i\alpha}), \tfrac{1}{2}(\bar{Q}_i^{\dot{\alpha}} + \bar{S}_i^{\dot{\alpha}}) \right\}, \\
\mathfrak{g}_{(2)} &= \left\{ \tfrac{1}{2}(P_a + K_a), D, R_{[ij]} \right\}, \\
\mathfrak{g}_{(3)} &= \left\{ \tfrac{-\mathrm{i}}{2}(Q^{i\alpha} - S^{i\alpha}), \tfrac{\mathrm{i}}{2}(\bar{Q}_i^{\dot{\alpha}} - \bar{S}_i^{\dot{\alpha}}) \right\}.
\end{aligned}$$

Choice of coset representative (H-gauge fixing)
adapted to Poincare form of metric

$$\mathrm{d}s^2 = -\tfrac{1}{2} Y^2 \mathrm{d}X_{\alpha\dot{\beta}} \mathrm{d}X^{\dot{\beta}\alpha} + \frac{1}{4Y^2}\, \mathrm{d}Y_{ij} \mathrm{d}Y^{ij}$$

$(X, Y) = (X^{\dot{\alpha}\beta}, Y^{ij})$ are 4+6 bosonic coordinates

$$\begin{aligned}
g(X, Y, \Theta) &= B(X, Y)\, \mathrm{e}^{-F(\Theta)}, \\
B(X, Y) &= \mathrm{e}^{\mathrm{i} X P}\, \mathrm{e}^{\mathrm{i}\log(Y)\, D}\, \Lambda(Y) \\
F(\Theta) &= \mathrm{i}\left[(\theta_+^{i\alpha} Q_{i\alpha} + \theta_-^{i\alpha} S_{i\alpha}) - (\bar{\theta}^{\dot{\alpha}}_{+i} \bar{Q}^i_{\dot{\alpha}} + \bar{\theta}^{\dot{\alpha}}_{-i} \bar{S}^i_{\dot{\alpha}}) \right], \\
\Lambda(Y) &= (\Lambda^i{}_j) = \tfrac{1}{Y}(C^{ik} Y_{kj}).
\end{aligned}$$

$\Theta = (\theta^{i\alpha}_{\pm}, \bar\theta^{\dot\alpha}_{\pm i})$ are 32 fermionic coordinates, $\theta^{i\alpha}_{\pm} \ = \ (\bar\theta^{\dot\alpha}_{\pm i})^{\dagger}$.

κ-gauge ("S-gauge") that simplifies structure of string action

$$\theta^{i\alpha}_{-} \ = \ \bar\theta^{\dot\alpha}_{-i} = 0, \qquad F(\Theta) \ = \ \mathrm{i}\big[\theta^{i\alpha}_{+} Q_{i\alpha} - \bar\theta^{\dot\alpha}_{+i}\bar Q^{i}_{\dot\alpha}\big]$$

Field redefinition:

$$(\theta^{i\alpha}_{+}, \bar\theta^{\dot\alpha}_{+i}) \ \mapsto \ (\theta^{i\alpha}, \bar\theta^{\dot\alpha}_{i}), \qquad \theta^{i\alpha} \ = \ Y^{-1/2}(\Lambda^{-1})^{i}{}_{j}\theta^{j\alpha}_{+}$$

Then string action (after a rotation of Y)

$$S \ = \int \Big\{ -\tfrac{1}{2} Y^2 \Pi_{\alpha\dot\beta} \wedge *\Pi^{\dot\beta\alpha} + \tfrac{1}{4Y^2}\, \mathrm{d}Y_{ij} \wedge *\mathrm{d}Y^{ij}$$
$$+ \tfrac{1}{2}\big(\epsilon_{\alpha\beta}\, \mathrm{d}Y_{ij} \wedge \theta^{i\alpha}\mathrm{d}\theta^{j\beta} - \epsilon_{\dot\alpha\dot\beta}\, \mathrm{d}Y^{ij} \wedge \bar\theta^{\dot\alpha}_{i}\mathrm{d}\bar\theta^{\dot\beta}_{j}\big)\Big\}$$

$$\Pi^{\dot\alpha\beta} \ = \ \mathrm{d}X^{\dot\alpha\beta} + \tfrac{\mathrm{i}}{2}(\bar\theta^{\dot\alpha}_{i}\mathrm{d}\theta^{i\beta} - \mathrm{d}\bar\theta^{\dot\alpha}_{i}\theta^{i\beta}).$$

Bosonic 2d duality along 4 X:

$$\int \Big[-\tfrac{1}{2} Y^2 \big(\mathrm{V}^{\dot\alpha\beta} + \tfrac{\mathrm{i}}{2}(\bar\theta^{\dot\alpha}_{i}\mathrm{d}\theta^{i\beta} - \mathrm{d}\bar\theta^{\dot\alpha}_{i}\theta^{i\beta})\big)^2 + \widetilde X_{\alpha\dot\beta}\mathrm{d}\mathrm{V}^{\dot\beta\alpha}$$
$$+ \tfrac{1}{4Y^2}\, \mathrm{d}Y_{ij} \wedge *\mathrm{d}Y^{ij} + \tfrac{1}{2}\big(\mathrm{d}Y_{ij} \wedge \theta^{i\alpha}\mathrm{d}\theta^{j}_{\alpha} - \mathrm{d}Y^{ij} \wedge \bar\theta^{\dot\alpha}_{i}\mathrm{d}\bar\theta_{j\dot\alpha}\big)\Big]$$

V - auxiliary one-form; $\widetilde{X}^{\alpha\dot{\beta}}$ imposes $\mathrm{dV}=0 \to \mathrm{V}=\mathrm{d}X$; solving for V first (Kallosh, AT 98)

$$\widetilde{S} = \int \Big\{ - \tfrac{1}{2Y^2} \mathrm{d}\widetilde{X}_{\alpha\dot{\beta}} \wedge *\mathrm{d}\widetilde{X}^{\dot{\beta}\alpha} + \tfrac{1}{4Y^2}\, \mathrm{d}Y_{ij} \wedge *\mathrm{d}Y^{ij}$$

$$+ \ \tfrac{\mathrm{i}}{2} \mathrm{d}\widetilde{X}_{\beta\dot{\alpha}} \wedge (\bar{\theta}^{\dot{\alpha}}_i \mathrm{d}\theta^{i\beta} - \mathrm{d}\bar{\theta}^{\dot{\alpha}}_i \theta^{i\beta}) + \tfrac{1}{2}(\mathrm{d}Y_{ij} \wedge \theta^{i\alpha}\mathrm{d}\theta^j_\alpha + c.c.)\Big\}.$$

(i) bosonic geometry is again $\mathrm{AdS}_5 \times S^5$ (up to $Y \mapsto Y^{-1}$)

(ii) the dual action is quadratic in the fermions

on-shell relation between the original and dual coordinates is

$$\mathrm{d}X^{\dot{\alpha}\beta} + \tfrac{\mathrm{i}}{2}(\bar{\theta}^{\dot{\alpha}}_i \mathrm{d}\theta^{i\beta} - \mathrm{d}\bar{\theta}^{\dot{\alpha}}_i \theta^{i\beta}) \ = \ Y^{-2} *\mathrm{d}\widetilde{X}^{\dot{\alpha}\beta}.$$

Can use it in the Noether currents of original model

$J_{\scriptscriptstyle N} \ = \ g\big[j_{(2)} - \tfrac{1}{2}*(j_{(1)} - j_{(3)})\big]g^{-1}$

to find their (non-local) expression in the dual model

Duality as symmetry of 1-st order system and Lax connection

How conserved charges of original and dual models are related?
duality applied to bosonic AdS_n-model generically maps conserved local charges into non-local ones and vice versa (Ricci, AT, Wolf 07)
first ignore fermions: back to bosonic $AdS_5 = SO(2,4)/SO(1,4)$
consider Z_2-automorphism of conformal $so(2,4)$ algebra

$$\Omega\big(P\big) = -K, \quad \Omega\big(K\big) = -P, \quad \Omega\left(D\right) = -D, \quad \Omega\left(L\right) = L$$

For choice of AdS_5 coset representative $g = \mathrm{e}^{\mathrm{i}XP} Y^{\mathrm{i}D}$

$$j = g^{-1}\mathrm{d}g = j_P + j_D, \quad j_P = \mathrm{i}Y\mathrm{d}X^{\dot{\alpha}\beta}P_{\beta\dot{\alpha}}, \quad j_D = \frac{\mathrm{i}}{Y}\mathrm{d}Y D$$

Then 1-st order system is

$$\begin{aligned}
\mathrm{d}j_P + j_D \wedge j_P + j_P \wedge j_D &= 0, \qquad \mathrm{d}j_D = 0, \\
\mathrm{d}{*}j_P - j_D \wedge {*}j_P - {*}j_P \wedge j_D &= 0, \\
\mathrm{d}{*}j_D - \tfrac{1}{2} j_P \wedge {*}\Omega\left(j_P\right) - \tfrac{1}{2}{*}\Omega\left(j_P\right) \wedge j_P &= 0.
\end{aligned}$$

under T-duality: $(X^{\dot\alpha\beta}, Y) \to (\widetilde{X}^{\dot\alpha\beta}, \widetilde{Y})$

$$\mathrm{d}\widetilde{X}^{\dot\alpha\beta} = Y^2 {*}\mathrm{d}X^{\dot\alpha\beta}\,, \qquad \widetilde{Y} = Y^{-1}.$$

$$j_P = \mathrm{i}Y\mathrm{d}X^{\dot\alpha\beta}P_{\beta\dot\alpha} = \mathrm{i}\widetilde{Y}{*}\mathrm{d}\widetilde{X}^{\dot\alpha\beta}P_{\beta\dot\alpha} = {*}\widetilde{j}_P,$$

$$j_D = \tfrac{\mathrm{i}}{Y}\mathrm{d}YD = -\tfrac{\mathrm{i}}{\widetilde{Y}}\mathrm{d}\widetilde{Y}D = -\widetilde{j}_D$$

this transformation, i.e.

$$j_P \mapsto \widetilde{j}_P = {*}j_P \qquad \text{and} \qquad j_D \mapsto \widetilde{j}_D = -j_D \qquad (*)$$

is symmetry of first-order equations

(MC equation for j_P is interchanged with its eq. of motion)

Thus can view it as a symmetry of phase space equations

regardless particular parametrization

Family of flat currents

$$\begin{aligned}\mathrm{j}(z) = {} & \tfrac{1}{4}(z+z^{-1})^2 j_P - \tfrac{1}{4}(z-z^{-1})^2\Omega(j_P) \\ & -\tfrac{1}{4}(z^2-z^{-2}){*}\big(j_P - \Omega(j_P)\big) + \tfrac{1}{2}(z^2+z^{-2})j_D \\ & -\tfrac{1}{2}(z^2-z^{-2}){*}j_D\end{aligned}$$

$\mathrm{j}(z)$ in the T-dual model should be the same

with $(X^{\dot\alpha\beta}, Y) \mapsto (\widetilde X^{\dot\alpha\beta}, \widetilde Y = Y^{-1})$

(*) gives apparently different result

$$
\begin{aligned}
\widetilde{\mathrm{j}}(z) \;=\; & \tfrac14 (z+z^{-1})^2 {*} j_P - \tfrac14 (z-z^{-1})^2 {*}\Omega(j_P) \\
& -\tfrac14 (z^2 - z^{-2})\left(j_P - \Omega(j_P)\right) - \tfrac12 (z^2+z^{-2}) j_D \\
& +\tfrac12 (z^2 - z^{-2}) {*} j_D
\end{aligned}
$$

But no doubling – two Lax connections are equivalent:

related by a Z_2-automorphism of $so(2,4)$:

$$
\mathcal{U}_z(T) \;=\; U_z \Omega(T) U_z^{-1}, \quad U_z \;=\; [f(z)]^{\mathrm{i}D}, \quad f = \frac{z - z^{-1}}{z + z^{-1}}
$$

$$
\mathcal{U}_z(j_P) = f(z)\Omega(j_P), \; \mathcal{U}_z(\Omega(j_P)) = (f(z))^{-1} j_P, \; \mathcal{U}_z(j_D) = -j_D,
$$

it maps the two Lax connections into each other

$$
\mathcal{U}_z(\mathrm{j}(z)) \;=\; \widetilde{\mathrm{j}}(z)
$$

Thus T-duality can be abstractly understood as
symmetry of the Lax connection (integrable structure)
induced by the automorphism of the conformal algebra $so(2,4)$
This symmetry then implies a certain map of conserved charges
Analogous automorphism once fermions are included?
κ-symmetry gauge choice makes some of
super-isometries non-manifest; transformed action not the same
add transformations of components of fermionic current
that will lead to symmetry of the full 1-st order GS system

Duality is an equivalence at the full $2d$ field theory level:
original global symmetry and its conserved charges
should not actually disappear but may become non-local
or hidden (not visible in the point-particle limit of the action)
to recover the original global symmetry

Bosonic+Fermionic duality: self-duality of superstring action

Combine bosonic duality with a similar fermionic one:
applying 2d duality to $\theta^{i\alpha}$ (but not to their conjugates $\bar{\theta}_i^{\dot{\alpha}}$).
Get action that can be interpreted as original $AdS_5 \times S^5$ superstring written in a different κ-symmetry gauge.
Thus combination of bosonic + fermionic dualities maps superstring action into an equivalent action.
Find full global superconformal group now acting (modulo a compensating κ-symmetry transformation) on coordinates of the dual action.
1-st order form of Lagrangian after bosonic duality:

$$-\frac{1}{2Y^2}\mathrm{d}\widetilde{X}_{\alpha\dot{\beta}} \wedge *\mathrm{d}\widetilde{X}^{\dot{\beta}\alpha} + \frac{1}{4Y^2}\,\mathrm{d}Y_{ij} \wedge *\mathrm{d}Y^{ij} - \mathrm{i}\widetilde{X}_{\beta\dot{\alpha}}\mathrm{d}\bar{\theta}_i^{\dot{\alpha}} \wedge \mathcal{V}^{i\beta}$$
$$-\frac{1}{2}Y_{ij}\mathcal{V}^{i\alpha} \wedge \mathcal{V}^j_\alpha - \widetilde{\theta}_{i\alpha} \wedge \mathrm{d}\mathcal{V}^{i\alpha} + \frac{1}{2}Y^{ij}\mathrm{d}\bar{\theta}_i^{\dot{\alpha}} \wedge \mathrm{d}\bar{\theta}_{j\dot{\alpha}}$$

constraint $\mathrm{d}\mathcal{V}^{i\alpha} = 0$ added with Lagrange multiplier $\widetilde{\theta}_{i\alpha}$

$$\mathcal{V}^{i\alpha} \; = \; -\tfrac{1}{Y^2} Y^{ij} \epsilon^{\alpha\beta} (\mathrm{d}\widetilde{\theta}_{j\beta} - \mathrm{i}\widetilde{X}_{\beta\dot{\alpha}} \mathrm{d}\bar{\theta}_j^{\dot{\alpha}}) = \mathrm{d}\theta^{i\alpha}$$

cf. bosonic duality: no Hodge star – fermions appear in WZ term

solve for $\mathcal{V}$: dual action for $\widetilde{\theta}$

$$- \tfrac{1}{2Y^2} \mathrm{d}\widetilde{X}_{\alpha\dot{\beta}} \wedge *\mathrm{d}\widetilde{X}^{\dot{\beta}\alpha} + \tfrac{1}{4Y^2} \, \mathrm{d}Y_{ij} \wedge *\mathrm{d}Y^{ij} + \tfrac{1}{2} Y^{ij} \mathrm{d}\bar{\theta}_i^{\dot{\alpha}} \wedge \mathrm{d}\bar{\theta}_{j\dot{\alpha}}$$
$$- \; \tfrac{1}{2Y^2} Y^{ij} \epsilon^{\alpha\beta} (\mathrm{d}\widetilde{\theta}'_{i\alpha} + \mathrm{i}\mathrm{d}\widetilde{X}_{\alpha\dot{\gamma}} \bar{\theta}_i^{\dot{\gamma}}) \wedge (\mathrm{d}\widetilde{\theta}'_{j\beta} + \mathrm{i}\mathrm{d}\widetilde{X}_{\beta\dot{\delta}} \bar{\theta}_j^{\dot{\delta}})$$

where $\widetilde{\theta}'_{i\alpha} = \; \widetilde{\theta}_{i\alpha} - \mathrm{i}\widetilde{X}_{\alpha\dot{\beta}} \bar{\theta}_i^{\dot{\beta}}$

Key point: this action is equivalent via field redefinition
to original $AdS_5 \times S^5$ GS action
in a different (complex) κ-gauge ($\bar{\text{Q}}$S-gauge)
(Roiban, Siegel '00)

$$\theta^{i\alpha}_{-} = 0, \quad \bar{\theta}^{\dot{\alpha}}_{+i} = 0$$

i.e. with coset representative $g = B(X, Y)\, \mathrm{e}^{-F(\Theta)}$

$$F(\Theta) = \mathrm{i}\big(\theta^{i\alpha}_{+} Q_{i\alpha} + \bar{\theta}^{\dot{\alpha}}_{-i} \bar{S}^{i}_{\dot{\alpha}}\big),$$

thus combination of bosonic and fermionic dualities relates
$AdS_5 \times S^5$ action in the κ-symmetry S-gauge
to same action in the κ-symmetry $\bar{\text{Q}}$S-gauge
implies existence of superconformal symmetry after the dualities
now explain the need for fermionic duality from
more general point of view: bosonic+fermionic dualities
leave superstring 1-st order system and Lax connection invariant

Bosonic+fermionic duality as symmetry of Lax connection

bosonic AdS_5 case: T-duality a symmetry of 1st-order system combined with a particular automorphism of conformal algebra
Now extend that symmetry to full superstring
by relating it to an automorphism of superconformal algebra.
Z_4 automorphism of $psu(2,2|4)$

$$\Omega(P_{\alpha\dot\beta}) = -K_{\alpha\dot\beta},\ \Omega(K_{\alpha\dot\beta}) = -P_{\alpha\dot\beta},\ \Omega(D) = -D,$$
$$\Omega(R_{[ij]}) = -R_{[ij]},\ \ \Omega(R_{(ij)}) = R_{(ij)},\ \Omega(Q^{i\alpha}) = \mathrm{i}S^{i\alpha},$$
$$\Omega(\bar Q_i^{\dot\alpha}) = \mathrm{i}\bar S_i^{\dot\alpha},\ \Omega(S_i^{\alpha}) = -\mathrm{i}Q_i^{\alpha},\ \Omega(\bar S^{i\dot\alpha}) = -\mathrm{i}\bar Q^{i\dot\alpha}$$

combined duality relation

$$\mathrm{d}X^{\dot\beta\alpha} + \tfrac{\mathrm{i}}{2}(\bar\theta_i^{\dot\beta}\mathrm{d}\theta^{i\alpha} - \mathrm{d}\bar\theta_i^{\dot\beta}\theta^{i\alpha}) = Y^{-2} *\mathrm{d}\widetilde X^{\dot\beta\alpha},$$
$$\mathrm{d}\theta^{i\alpha} = -\tfrac{1}{Y^2}Y^{ij}\epsilon^{\alpha\beta}(\mathrm{d}\widetilde\theta_{j\beta} - \mathrm{i}\widetilde X_{\beta\dot\alpha}\mathrm{d}\bar\theta_j^{\dot\alpha}),\quad \widetilde Y = Y^{-1}$$

relate current in S-gauge $j = j_P + j_D + j_R + j_Q + j_{\bar Q}$

to dual one in the $\bar{\mathrm{Q}}$S-gauge $\widetilde{j} = \widetilde{j}_P + \widetilde{j}_D + \widetilde{j}_R + \widetilde{j}_Q + \widetilde{j}_{\bar{S}}$

$$\widetilde{j}_P = *j_P, \;\; \widetilde{j}_D = -j_D, \;\; \widetilde{j}_{R_a} = -j_{R_a}, \;\; \widetilde{j}_{R_s} = j_{R_s},$$
$$\widetilde{j}_Q = j_Q, \quad \widetilde{j}_{\bar{S}} = -\mathrm{i}\Omega(j_{\bar{Q}}).$$

Flat currents: in the S-gauge

$$\mathrm{j}(z) = \mathrm{j}_B(z) + \tfrac{1}{2}(z+z^{-1})(j_Q+j_{\bar{Q}}) - \tfrac{\mathrm{i}}{2}(z-z^{-1})(\Omega(j_Q)+\Omega(j_{\bar{Q}})),$$

the dual one in $\bar{\mathrm{Q}}$S-gauge

$$\widetilde{\mathrm{j}}(z) = \widetilde{\mathrm{j}}_B(z) + \tfrac{1}{2}(z+z^{-1})(j_Q-\mathrm{i}\Omega(j_{\bar{Q}})) + \tfrac{\mathrm{i}}{2}(z-z^{-1})(\Omega(j_Q)+\mathrm{i}j_{\bar{Q}}),$$

are related by a Z_4 automorphism of the superconformal algebra:

$$\mathcal{U}_z(T) = U_z\,\Omega(T)\,U_z^{-1}, \quad U_z = \mathrm{e}^{-\pi\mathrm{B}}(f(z))^{\mathrm{i}(\mathrm{B}+D)}$$

$f(z) = \frac{z-z^{-1}}{z+z^{-1}}$ and $[\mathrm{B}, Q] = \frac{\mathrm{i}}{2}Q, \;\; [\mathrm{B}, S] = -\frac{\mathrm{i}}{2}S$, etc.

Explicitly

$$\mathcal{U}_z(P) = f(z)\Omega(P),\ \mathcal{U}_z(K) = f^{-1}(z)\Omega(K),\ \mathcal{U}_z(D) = \Omega(D),$$
$$\mathcal{U}_z(Q^{i\alpha}) = \mathrm{i}f(z)\Omega(Q^{i\alpha}), \qquad \mathcal{U}_z(S_i^{\alpha}) = -\mathrm{i}f(z)^{-1}\Omega(S_i^{\alpha}),$$
$$\mathcal{U}_z(\bar{Q}_i^{\dot{\alpha}}) = -\mathrm{i}\Omega(\bar{Q}_i^{\dot{\alpha}}),\ \ \mathcal{U}_z(\bar{S}^{i\dot{\alpha}}) = \mathrm{i}\Omega(\bar{S}^{i\dot{\alpha}}).$$

then Lax connections are related as

$$\widetilde{\mathrm{j}}(z) = \mathcal{U}_z(\mathrm{j}(z))$$

i.e. duality is symmetry of integrable structure
and 1-st order system
conserved charges are not doubled but reshuffled

Noether charges may be derived from flat current $\mathrm{j}(z)$ at $z = \pm 1$:
superconformal Noether charges ($f \to 0,\ z \to \pm 1$) behave as

- $P_{\alpha\dot{\beta}}$-charge becomes trivial
- $L_{\alpha\beta}$- and $L_{\dot{\alpha}\dot{\beta}}$-charges go into themselves and thus local
- $K_{\alpha\dot{\beta}}$-charge gets lifted and becomes non-local
- D-charge goes into itself and thus remains local
- $R_i{}^j$-charge goes into itself and thus remains local
- $Q^{i\alpha}$-charge becomes trivial
- $\bar{Q}_i^{\dot{\alpha}}$-charge goes into the $\bar{S}^{i\dot{\alpha}}$-charge and thus remains local
- S_i^{α}-charge gets lifted and becomes non-local
- $\bar{S}^{i\dot{\alpha}}$-charge goes into the $\bar{Q}_i^{\dot{\alpha}}$-charge and thus remains local

$P_{\alpha\dot{\beta}}$ and $Q^{i\alpha}$ do not act on dual fields $\widetilde{X}_{\alpha\dot{\beta}}$ and $\widetilde{\theta}_{i\alpha}$

resulting picture in agreement with
parallel work of Berkovits and Maldacena 08

similar relations for the generators of the original and dual
superconformal symmetry when acting on supergluon amplitudes
(Drummond, Henn, Korchemsky, Sokatchev 08)